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SOJOURNS, EXTREMES, and SELF-INTERSECTIONS of STOCHASTIC PROCESSES

FINAL TECHNICAL REPORT

SIMEON M. BERMAN

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NEW YORK UNIVERSITY
COURANT INSTITUTE OF MATHEMATICAL SCIENCES
251 Mercer Street
New York, N.Y. 10012

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Simeon M. Berman

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251 Mercer Street
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13. ABSTRACT *(Maximum 200 words)*

The subjects of this research are certain probabilistic properties of real and vector valued random functions X(t), where t is a generalized time parameter taking values in a subset T of Euclidean space. The distributions of three functionals of the random function X are studied. For any Borel set A in the range, the sojourn time of X in A is defined as the measure of the subset of the points t in T such that X(t) belongs to A. The self-intersection set of X is the subset of the set of points (s,t) in the product space of T such that s = t and X(s) = X(t). Finally, for a real valued random function X, the extreme value is the functional equal to the maximum value of X on the domain T. The research is concerned with the determination of the distributions of these functionals under various hypotheses about the probabilistic structure of X. It is assumed here that the random function is Gaussian or Markovian.

14. SUBJECT TERMS Stochastic process, extreme value, sojourn time, local time, limiting distribution, self-intersections of paths, Gaussian process, Markov process.

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Spectral conditions for sojourn and extreme value limit theorems for Gaussian processes

Simeon M. Berman
Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA

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Let $X(t), t \geq 0$, be a stationary Gaussian process, and define the sojourn time $L_u(t) = \text{mes}\{s: 0 \leq s \leq t, X(s) > u\}$ and the maximum $Z(t) = \max(X(s): 0 \leq s \leq t)$. Limit theorems for the distributions of $L_u(t)$ and $Z(t)$, for $t, u \to \infty$, are obtained under specified conditions on the spectral density of the process. The results supplement earlier theorems obtained under suitable conditions on the covariance function.

AMS 1980 Subject Classifications: 60F05, 60G10, 60G15.

stationary Gaussian processes * spectral density function * mixing condition * sojourn above a level * extreme value

1. Introduction and summary

Let $X(t), t \geq 0$, be a stationary Gaussian process with mean 0, variance 1 and covariance function $r(t) = EX(0)X(t)$. There is now an extensive literature on the functionals of this process which are associated with the large values of the sample functions. Key examples of such functionals are (i) the sojourn time above a high level u,

$$L_u(t) = \int_0^t 1_{[X(s)>u]}\, ds; \tag{1.1}$$

and (ii) the maximum value

$$Z(t) = \max_{0 \leq s \leq t} X(s). \tag{1.2}$$

The latter is well defined because the conditions assumed in the hypotheses of the theorems imply the continuity of the sample functions. Results of much mathematical interest are limit theorems for the distributions of these functionals for $t \to \infty$, $u \to \infty$, where t and u are tied together by means of a specified asymptotic relation. Among the recent results for $L_u(t)$ are those of Berman (1980, 1989). Surveys of results for $Z(t)$ are those of Leadbetter et al. (1983) and Leadbetter and Rootzen (1988).

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Central Limit Theorems for Extreme Sojourns of Stationary Gaussian Processes

SIMEON M. BERMAN
Courant Institute

This paper is dedicated to Natascha A. Brunswick for years of talented service, dedication, and good cheer.

Abstract

Let $X(t)$, $t \geqq 0$, be a real stationary Gaussian process, and, for $u > 0$ and $t > 0$, let $L_t(u)$ be the time spent by $X(s)$, $0 \leqq s \leqq t$, above the level u. Here u is taken to be a function $u(t)$ of t, and L_t is defined as $L_t(u(t))$. It is shown that the distribution of $(L_t - EL_t)/(\text{Var } L_t)^{1/2}$ converges, for $t \to \infty$ and $u(t) \to \infty$, to a standard normal distribution under various conditions relating the growth of $u(t)$ to the decay of the covariance and other functions associated with it.

1. Introduction and Summary

Let $X(t)$, $t \geqq 0$, be a real measurable stationary Gaussian process with mean 0 and covariance function $r(t) = EX(0)X(t)$. For simplicity we take $r(0) = 1$. For $t > 0$, put $L_t(u) = \text{mes}(s : 0 \leqq s \leqq t, X(s) > u)$; then, for a given measurable function $u(t)$, we define

$$L_t = L_t(u(t)) = \int_0^t 1_{[X(s) > u(t)]}\, ds . \tag{1.1}$$

The main results of this paper are several new central limit theorems for the distribution of $(L_t - EL_t)/(\text{Var } L_t)^{1/2}$, for $t \to \infty$, under various conditions on the covariance function and the function $u(t)$. In this work we assume that the spectral distribution function in the representation of $r(t)$ is absolutely continuous. Let $f(\lambda)$ be the spectral density function, and let $b(t)$ be the Fourier transform of the L_2-function $(f(\lambda))^{1/2}$; then $r(t)$ has the representation

$$r(t) = \int_{-\infty}^{\infty} b(t+s) b(s)\, ds , \tag{1.2}$$

and $b \in L_2$. Furthermore, $X(t)$ has the stochastic integral representation

$$X(t) = \int_{-\infty}^{\infty} b(t+s) \xi(ds) , \tag{1.3}$$

where $\xi(s)$ is a standard Brownian motion.

The focal point of the calculations in the proofs of these theorems is the relation

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$(2/u)^{1/2} \to 0$, which proves (6.7). In [2] conditions (1.5) and (1.7) above are used to prove that $v(t) \to \infty$ and $\exp\{\delta u^2\}\beta(v/2) \to 0$. We will show that the latter relations remain true under (1.5) and (1.7) with the new definition (7.1) of v. Indeed, if (1.5) holds for some $\theta > 1$, then, for every θ', with $1 < \theta' < \theta$, by the relation (5.6) of [2] with θ' in the place of θ,

$$v \geqq \text{constant}(t/u^2)^{1/2}\exp\{-u^2\theta'/4\}$$

$$= \text{constant } t^{1/2}\exp\{-u^2\theta/4\}\,\frac{\exp\{(u^2/4)(\theta-\theta')\}}{u},$$

from which it follows by (1.5) that $v \to \infty$ and by (1.7) that $\exp\{\delta u^2\}\beta(v/2) \to 0$. In the relation (5.9) of [2] the factor $u^{1/4}$ should be changed to $u^{-1/2}$.

I thank Katarzina Pietruska-Paluba for noticing this error.

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A CENTRAL LIMIT THEOREM FOR INTEGRAL FUNCTIONALS OF A STATIONARY GAUSSIAN PROCESS*

Simeon M. Berman

Courant Institute of Mathematical Sciences

New York University

New York, NY 10012

ABSTRACT. Let $X(t)$, $t \geq 0$, be a real stationary Gaussian process with covariance function $r(t)$. Let $f(x)$ be a function in $L_2(\phi)$, where $\phi(z)$ is the standard normal density, and assume that $\int x\, f(x)\, \phi(x)\, dx \neq 0$. It is shown that the central limit theorem holds for the functional $\int_0^t f(X(s))\, ds$, for $t \to \infty$, under the sole assumptions $r(t) \geq 0$ and $r(t) \to 0$ for $t \to \infty$.

1. Summary.

Let $X(t)$, $t \geq 0$, be a real stationary Gaussian process with mean 0 and covariance function $r(t) = EX(0)\, X(t)$. For a Borel function $f(x)$ and $t > 0$, consider the functional

$$\int_0^t f(X(s))\, ds\ . \tag{1.1}$$

There has been a sustained interest in proving the Central Limit Theorem, for $t \to \infty$, for such functionals, that is, determining the limiting distribution of the normed random variable

$$\frac{\int_0^t f(X(s))\, ds - E[\int_0^t f(X(s))\, ds]}{\{Var \int_0^t f(X(s))\, ds\}^{1/2}}\ . \tag{1.2}$$

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A CENTRAL LIMIT THEOREM FOR THE RENORMALIZED SELF-INTERSECTION LOCAL TIME OF A STATIONARY VECTOR GAUSSIAN PROCESS[1]

By Simeon M. Berman

New York University

Let $\mathbf{X}(t)$ be a stationary vector Gaussian process in R^m whose components are independent copies of a real stationary Gaussian process with covariance function $r(t)$. Let $\phi(z)$ be the standard normal density and, for $t > 0$, $\varepsilon > 0$, consider the double integral

$$\int_0^t \int_0^t \varepsilon^{-m} \prod_{j=1}^m \phi\big(\varepsilon^{-1}(X_j(s) - X_j(s'))\big)\, ds\, ds',$$

which represents an approximate self-intersection local time of $\mathbf{X}(s)$, $0 \le s \le t$. Under the sole condition $r \in L_2$, the double integral has, upon suitable normalization, a limiting normal distribution under a class of limit operations in which $t \to \infty$ and $\varepsilon = \varepsilon(t)$ tends to 0 sufficiently slowly. The expected value and standard deviation of the double integral, which are the normalizing functions, are asymptotically equal to constant multiples of t^2 and $t^{3/2}$, respectively. These results are valid without any restrictions on the behavior of $r(t)$ for $t \to 0$ other than continuity.

1. Introduction and summary. Let $X(t)$, $t \ge 0$, be a real, measurable stationary Gaussian process. For simplicity, take $EX(t) = 0$ and $EX^2(t) = 1$ and let $r(t) = EX(0)X(t)$ be the covariance function, which is assumed to be continuous. For $m \ge 1$, let $X_1(t), \dots, X_m(t)$, $t \ge 0$, be independent copies of $X(t)$, and define the vector process $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))$. Put

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right), \tag{1.1}$$

and, for $\varepsilon > 0$ and $t > 0$, consider the random variable

$$\varepsilon^{-m} \int_0^t \int_0^t \prod_{j=1}^m \phi\left(\frac{X_j(s) - X_j(s')}{\varepsilon}\right) ds\, ds'. \tag{1.2}$$

The following theorem is our main result.

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Key words and phrases. Central limit theorem, mixing, renormalized local time, self-intersections, stationary Gaussian process.

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THE TAIL OF THE CONVOLUTION OF DENSITIES AND ITS APPLICATION TO A MODEL OF HIV-LATENCY TIME[1]

BY SIMEON M. BERMAN

New York University

Let $p(x)$ and $q(x)$ be density functions and let $(p * q)(x)$ be their convolution. Define

$$w(x) = -(d/dx)\log q(x) \quad \text{and} \quad v(x) = -(d/dx)\log p(x).$$

Under the hypothesis of the regular oscillation of the functions w and v, the asymptotic form of $(p * q)(x)$, for $x \to \infty$, is obtained. The results are applied to a model previously introduced by the author for the estimation of the distribution of HIV latency time.

1. Introduction and summary. Let $p(x)$ and $q(x)$ be probability density functions and $(p * q)(x)$ their convolution. The focus of this paper is the determination of the asymptotic form of $(p * q)(x)$ for $x \to \infty$ or $x \to b = \sup$ (support of $p * q$) on the basis of the asymptotic forms of $p(x)$ and $q(x)$. In most of this paper p and q are assumed to be of different orders of magnitude for $x \to \infty$ and q will have the role of the density with the heavier tail. The key tools for densities p and q with unbounded support are the functions $v(x) = -(d/dx)\log p(x)$ and $w(x) = -(d/dx)\log q(x)$, which are closely related to the hazard functions used in extreme value theory. The tail of q obviously dominates the tail of p whenever the reverse holds for their corresponding functions w and v. Throughout this paper it is assumed that $v(x)$ and $w(x)$ are nonnegative for all sufficiently large x. In particular, it follows that the corresponding densities are nonincreasing for such x.

Theorem 3.1 states that if $\limsup w(x) < \liminf v(x)$ for $x \to \infty$, then

$$\int_{-\infty}^{\infty} p(x-t)q(t)\,dt \sim q(x)\int_{-\infty}^{\infty} e^{tw(x)}p(t)\,dt.$$

This represents an extension of a corresponding result of Breiman (1965) and Cline (1986), stated in terms of distributions instead of densities for the case $w(x) \to c \geq 0$. Theorem 3.2 furnishes a general result under the condition $w(x)/v(x) \to 0$, which allows for even the cases $w(x) \to \infty$ or $v(x) \to 0$. It

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AMS 1980 *subject classifications*. Primary 60E99, 60F05; secondary 92A15.

Key words and phrases. Tail of a density function, convolution, regular oscillation, regular variation, extreme value distribution, domain of attraction, HIV latency time.

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A CENTRAL LIMIT THEOREM FOR THE RENORMALIZED SELF-INTERSECTION LOCAL TIME OF A STATIONARY PROCESS

Simeon M. Berman*

1. Introduction and Summary

Let $X(t)$, $t \geq 0$, be a measurable, separable stochastic process in R^m, for some $m \geq 1$. Let $\phi(x)$, $x \in R^m$, be a probability density function such that $\phi(x) = \phi(-x)$, and let $\hat{\phi}(u)$ be the corresponding characteristic function. It is assumed that

$$\int_{R^m} |\hat{\phi}(u)|\, du < \infty \, . \tag{1.1}$$

Let B be a symmetric ($B = -B$) closed bounded Borel set in R^m. For $\epsilon > 0$ and $t > 0$ define the functional

$$I_\epsilon(t) = \int_0^t \int_0^t \epsilon^{-m} \phi\left(\frac{X(s) - X(s')}{\epsilon}\right) 1_B(X(s))\, 1_B(X(s'))\, ds\, ds' \, ; \tag{1.2}$$

this is the approximate self-intersection local time of $X(s)$, $0 \leq s \leq t$, relative to B, for small $\epsilon > 0$. (The ratio $X(s) - X(s')/\epsilon$ in (1.2) is understood as ϵ^{-1} times the vector $X(s) - X(s')$.) This is a more general version of the functional introduced by the author (Berman, 1992) in the context of a stationary Gaussian process $X(t)$ in the special case where $B = R^m$ and ϕ is the m-dimensional product standard normal density. The inclusion of the indicator 1_B in the integrand restricts the approximate self-intersections to those for which the sample function values belong to B. In the previous work it was shown that if $X(t)$, $t \geq 0$, is a stationary Gaussian process in R^m whose components are independent copies of a real stationary Gaussian process, then, under a mild condition on the covariance function, the random variable $t^{-3/2}(I_\epsilon(t) - EI_\epsilon(t))$ has, under an appropriate class

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of limit operations $t \to \infty$, $\epsilon = \epsilon(t) \to 0$, a limiting $N(0, \sigma^2)$ distribution, where σ^2 is explicitly given. The current work extends this central limit theorem to the functional (1.2) and where the class of stochastic processes $X(t)$ includes, in addition to the Gaussian process studied in Berman (1992), a large class of ergodic stationary Markov processes.

Our main result is

THEOREM 1.1. *Let* $\mathrm{X}(t)$ *be a stationary vector process in* R^m *with marginal density* $p(x)$, $x \in R^m$. *Let* $p(x, y; t)$ *represent the joint density of* $X(0)$ *and* $X(t)$ *at* (x, y), *for* $t > 0$. *Let* $\epsilon = \epsilon(t)$ *be a decreasing function for* $t > 0$ *with* $\epsilon(t) \to 0$ *for* $t \to 0$, *where the convergence is so slow that*

$$\lim_{t\to\infty} t^{-1/2}(\epsilon(t))^{-m} \int_0^1 \int_B \int_B \phi\left(\frac{x-y}{\epsilon(t)}\right) p(x, y; s)\, dx\, dy\, ds = 0 \,. \tag{1.3}$$

Assume:

$$p(x) \text{ is continuous on } B \,, \tag{1.4}$$

$$p(x) = p(-x)\,, \qquad x \in B\,, \tag{1.5}$$

$$\int_c^\infty |p(x, y; s) - p(x)p(y)|\, ds < \infty \,, \qquad x, y \in B\,; \tag{1.6}$$

and that for every $c > 0$,

$$\int_c^\infty [p(x, y; s) - p(x)p(y)]\, ds \tag{1.7}$$

is continuous for $(x, y) \in B \times B$. *Assume also that the CLT holds for the integral functional* $\int_0^t 1_B(X(s))\,p(X(s))\,ds$ *in the sense*

$$t^{-1/2}\Big[\int_0^t 1_B(X(s))\,p(X(s))\,ds - t\int_B p^2(x)\,dx\Big] \xrightarrow{d} N(0, \sigma^2)\,, \tag{1.8}$$

for some σ^2, $0 < \sigma^2 < \infty$. *Then we conclude that*

$$\frac{I_{\epsilon(t)}(t) - E\, I_{\epsilon(t)}(t)}{t^{3/2}} \xrightarrow{d} N(0, 4\sigma^2)\,, \tag{1.9}$$

for $t \to \infty$. In this case σ^2 is given by

$$\sigma^2 = 2 \int_B \int_B p(x)\, p(y) \int_0^\infty \left[p(x,y;s) - p(x)\, p(y)\right] ds\, dx\, dy \ . \tag{1.10}$$

We note that the condition

$$t^{1/2}\, \epsilon^m(t) \to \infty \tag{1.11}$$

is sufficient for (1.3) because ϕ is bounded under (1.1) and $\int\int p(x,y;s)\, dx\, dy = 1$ for every $s > 0$. We also note that the scaling factor $t^{3/2}$ in (1.9) is the same for all dimensions $m \geq 1$.

For $\epsilon \to 0$ the integral (1.2) measures the set of points (s, s') where $X(s)$ and $X(s')$ are close. This includes both points near the diagonal and points bounded away from it. The latter are the "genuine" near-self-intersection points. We show that the near-diagonal points make an asymptotically negligible contribution to $I_\epsilon(t)$ in the statement of Theorem 4.1 in the sense that (1.9) still holds after the removal of these points from the domain of integration in (1.2). For this purpose, we prove:

THEOREM 1.2. *Under the conditions of Theorem* 1.1,

$$\begin{aligned} &\lim_{t\to\infty} t^{-3/2} E\Big(\int_0^t \int_0^t 1_{[0,1]}(|s - s'|) \\ &\epsilon^{-m}\, \phi\Big(\frac{X(s) - X(s')}{\epsilon}\Big)\, 1_B(X(s))\, 1_B(X(s'))\, ds\, ds'\Big) = 0 \ . \end{aligned} \tag{1.12}$$

PROOF. The expected value in (1.12) is at most equal to (see Section 6)

$$\begin{aligned} &2 \int_0^t \int_s^t 1_{[0,1]}(s' - s)\, \epsilon^{-m} \int_B \int_B \phi\Big(\frac{x - y}{\epsilon}\Big)\, p(x,y;s' - s)\, dx\, dy\, ds'\, ds \\ &\leq 2t \int_0^1 \epsilon^{-m} \int_B \int_B \phi\Big(\frac{x - y}{\epsilon}\Big)\, p(x,y;s)\, dx\, dy\, ds \ , \end{aligned}$$

which, under the condition (1.3) is of order smaller than $t^{3/2}$. □

The application of Theorem 1.1 to Markov processes $X(t)$ is demonstrated in Section 5. The bivariate density $p(x, y; s)$ assumes the particular form $p(x)\, q(s; x, y)$, where q is the transition density. The hypotheses (1.4)–(1.7) can be stated in terms

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A CENTRAL LIMIT THEOREM FOR EXTREME SOJOURNS OF DIFFUSION PROCESSES

Simeon M. Berman

Courant Institute of Mathematical Sciences

251 Mercer Street

New York, NY 10012

ABSTRACT

Let $X(t), t \geq 0$, be a real-valued diffusion process having a stationary probability measure. For an increasing function $u(t)$, $t > 0$, put $L(t) = mes\ (s{:}0 \leq s \leq t, X(s) > u(t))$. It is shown, under general conditions on the diffusion coefficients, that if $u(t) \to \infty$ at a sufficiently slow rate, then $(L(t) - EL(t))/(VarL(t))^{1/2}$ has, for $t \to \infty$, a limiting normal distribution. The rate of increase of $u(t)$ is stated in terms of the scale function $S(x)$ associated with the generator of the process; $u(t)$ must satisfy $S(u(t)) = o(t)$, for $t \to \infty$. This complements an earlier result (Berman, 1988) in the case $S(u(t)) \sim t$, where it was shown that there is a function $v(t)$ such that $v(t)L(t)$ has a particular limiting compound Poisson distribution.

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